Well-posedness and large deviations for 2D Stochastic Navier-Stokes equations driven by multiplicative Lévy noise

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This talk mainly based on

- Z. Brzeźniak, X. Peng, J. Zhai, Well-posedness and large deviations for 2-D Stochastic Navier-Stokes equations with jumps, *Journal of the European Mathematical Society*, 25, 3093–3176, 2023.

- Compared with the case of Gaussian noise, SPDEs driven by jump type noise such as Lévy-type or Poisson-type perturbations are drastically different because of the appearance of the jump.
    - stopping time
    - BDG inequality
    - well-posedness
    - time regularity
    - ergodic property
    - LDP, etc.

Outline

Introduction

Consider the two-dimensional Navier-Stokes equation

$$\frac{\partial u(t)}{\partial t} - \nu \Delta u(t) + (u(t) \cdot \nabla) u(t) + \nabla p(t,x) = f(t),$$

with the conditions

$$\begin{cases} (\nabla \cdot u)(t,x) = 0, & \text{for } \ x \in D, t > 0, \\ u(t,x) = 0, & \text{for } \ x \in \partial D, t \geq 0, \\ u(0,x) = u_0(x), & \text{for } \ x \in D, \end{cases}$$

where D is a bounded open domain of $\mathbb{R}^2$ with regular boundary ∂D, $u(t,x) \in \mathbb{R}^2$ denotes the velocity field at time t and position x, $\nu > 0$ is the viscosity, $p(t,x)$ denotes the pressure field, f is a deterministic external force.

Introduction

To formulate the Navier-Stokes equation, we introduce the following standard spaces: let

$$V = \left\{ v \in H_0^1(D; \mathbb{R}^2) : \nabla \cdot v = 0, \text{a.e. in } D \right\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v|^2 dx \right)^{\frac{1}{2}} = \|v\|,$$

and let H be the closure of V in the L^2-norm

$$|v|_H := \left(\int_D |v|^2 dx \right)^{\frac{1}{2}} = |v|.$$

Introduction

Define the operator A (Stokes operator) in H by the formula

$$Au := -P_H \Delta u, \quad \forall u \in H^2(D;\mathbb{R}^2) \cap V,$$

where the linear operator P_H (Helmhotz-Hodge projection) is the projection operator from $L^2(D;\mathbb{R}^2)$ to H,

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where the linear operator P_H (Helmhotz-Hodge projection) is the projection operator from $L^2(D;\mathbb{R}^2)$ to H, and define the nonlinear operator B by

$$B(u,v) := P_H((u \cdot \nabla)v),$$

with the notation $B(u) := B(u,u)$ for short.

Introduction

By applying the operator P_H to each term of (5), we can rewrite it in the following abstract form:

$$du(t) + Au(t)dt + B(u(t))dt = f(t)dt \quad \text{in } L^2([0,T], V'),$$

with the initial condition $u(0) = u_0$ for some fixed point u_0 in H.

Well-Posedness

Part 1. Well-posed for 2-D SNSEs driven by multiplicative Levy noise

Well-Posedness

- Given a probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, satisfying the usual condition,
- Z is a locally compact Polish space, ν is a σ-finite measure on Z
- Leb_∞ is the Lebesgue measure on $[0, \infty)$
- η is a Poisson random measure on $[0, \infty) \times Z$ with a σ-finite intensity measure $\mathrm{Leb}_\infty \otimes \nu$
- the compensated Poisson random measure

$$\widetilde{\eta}([0,t]\times O) = \eta([0,t]\times O) - t\nu(O), \ \forall O \in \mathcal{B}(Z) : \ \nu(O) < \infty.$$

Well-Posedness

We consider the following stochastic Navier-Stokes equations driven by multiplicative Lévy noise

$$du(t) + Au(t)\,dt + B(u(t))\,dt = f(t)\,dt + \int_Z G(u(t-), z)\widetilde{\eta}(dz, dt),$$
$$u_0 \in H.$$

Problem: Well-Posedness?

Well-Posedness

The Well-Posedness of strong solutions in probability sense

Well-Posedness

Assumption 1:

We assume that $G : \mathrm{H} \times \mathrm{Z} \to \mathrm{H}$ is a measurable map such that

for every $\hbar > 0$, there exists a constant $C_\hbar > 0$ such that, for all $v_1, v_2 \in \mathrm{H}$ with $|v_1|_{\mathrm{H}} \vee |v_2|_{\mathrm{H}} \leq \hbar$,

$$\int_{\mathrm{Z}} |G(v_1, z) - G(v_2, z)|_{\mathrm{H}}^2 \nu(dz) \leq C_\hbar |v_1 - v_2|_{\mathrm{H}}^2,$$

and it satisfies the Linear growth assumption, i.e.,

$$\int_{\mathrm{Z}} \|G(v, z)\|_{\mathrm{H}}^2 \nu(dz) \leq C(1 + \|v\|_{\mathrm{H}}^2), \quad \forall v \in \mathrm{H}.$$

Well-Posedness

Our First Result

Assume that **Assumption 1** holds. Then for every $u_0 \in \mathrm{H}$ and $f \in L^2_{loc}([0,\infty), \mathrm{V}')$ there exists a unique $\mathbb{F}$-progressively measurable process u such that

(1) $u \in D([0,\infty), H) \cap L^2_{loc}([0,\infty), V)$, $\mathbb{P}$-a.s.,

(2) the following equality holds, for all $t \in [0,\infty)$, $\mathbb{P}$-a.s., in V',

$$\begin{aligned} u(t) &= u_0 - \int_0^t Au(s)\,ds - \int_0^t B(u(s))\,ds + \int_0^t f(s)\,ds \\ &\quad + \int_0^t \int_Z G(u(s-), z)\widetilde{\eta}(dz, ds). \end{aligned}$$

Well-Posedness

The existing results

The existing results in the literature always need other assumptions on G, besides **Assumption 1**. For example,

- Z. Dong, Y. Xie, (2009) There exist measurable subsets U_m, $m \in \mathbb{N}$ of Z with $U_m \uparrow \mathrm{Z}$ and $\nu(U_m) < \infty$ such that, for some $k > 0$,

$$\sup_{\|v\|_{\mathrm{H}} \leq k} \int_{U_m} \|G(v,z)\|_{\mathrm{H}}^2 \nu(dz) \to 0 \text{ as } m \to \infty,$$

- Z. Brzeźniak, E. Hausenblas, J. Zhu, (2013); Z. Brzeźniak, W. Liu, J. Zhu, (2014) It is assumed that

$$\int_{\mathrm{Z}} \|G(v,z)\|_{\mathrm{H}}^4 \nu(dz) \leq K(1 + \|v\|_{\mathrm{H}}^4).$$

- E. Motyl(2014) They assume that for each $p \in \{1, 2, 2+\gamma, 4, 4+2\gamma\}, \gamma > 0$, there exists a constant $c_p > 0$ such that

$$\int_{\mathrm{Z}} \|G(v,z)\|_{\mathrm{H}}^p \nu(dz) \leq c_p(1 + \|v\|_{\mathrm{H}}^p).$$

Key points in the Proof.

Introduce the following notations

- For $T \geq 0$,

$$\Lambda_T = D([0,T],H) \cap L^2([0,T],\mathrm{V}).$$

The space Λ_T endowed with the norm

$$\|y\|^2_{\Lambda_T} = \mathbb{E}\Big[\sup_{s\in[0,T]} \|y(s)\|^2_{\mathrm{H}} + \int_0^T \|y(s)\|^2_{\mathrm{V}}\, ds\Big]$$

is a Banach space.

- For every $m \in \mathbb{N}\setminus\{0\}$, $\theta_m : [0,\infty) \to [0,1]$ satisfying

$$\begin{cases} \theta_m \in C^2[0,\infty); \\ \sup_{t\in[0,\infty)} |\theta'_m(t)| \leq C_1 < \infty; \\ \theta_m(t) = 1, \quad t \in [0,m]; \\ \theta_m(t) = 0, \quad t \geq m+1 \\ \theta_m(t) \in [0,1], \quad m < t < m+1 \end{cases}$$

where C_1 is m independent.

Key points in the Proof.

- Let us put, for $T \geq 0$,

$$\Upsilon_T(H) = D([0,T],H) \cap L^2([0,T],\mathrm{V}).$$

It is standard that the space $\Upsilon_T(H)$ endowed with the norm

$$\|y\|_{\Upsilon_T(H)} = \sup_{s\in[0,T]} \|y(s)\|_{\mathrm{H}} + \left(\int_0^T \|y(s)\|_{\mathrm{V}}^2\, ds\right)^{1/2}$$

Key points in the Proof.

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(1). For any $y \in \Lambda_T(\mathrm{H})$, Set

$$dM(t) + \mathrm{A}M(t)\,dt = \int_{\mathrm{Z}} G(y(t-), z)\widetilde{\eta}(dz, dt),$$
$$M(0) = 0.$$

(2). $\{y_n, n \in \mathbb{N}\}$ is a cauchy sequence in $C([0,T],H) \cap L^2([0,T],V)$, where

$$y'_{n+1}(t) + \mathrm{A}y_{n+1}(t) + \theta_m(\|y_n + M\|_{\Upsilon^H_t})\phi_\delta(\|y_n + M\|_{L^2([0,t];\mathrm{V})})$$
$$B(y_n(t) + M(t), y_{n+1}(t) + M(t)) = f(t),\ \ t \in [0,T];$$
$$y_{n+1}(0) = u_0.$$

Key points in the Proof.

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This implies the existence of the following deterministic PDEs

$$X'(t) + \mathrm{A}X(t) + \theta_m(\|X+M\|_{\Upsilon_t^H})\phi_\delta(\|X+M\|_{L^2([0,t];\mathrm{V})})B(X(t)+M(t))$$
$$= f(t),$$
$$X(0) = u_0.$$

and

$$X'(t) + \mathrm{A}X(t) + \theta_m(\|X+M\|_{\Upsilon_t^H})B(X(t)+M(t))$$
$$= f(t),$$
$$X(0) = u_0.$$

(3). By (1) and (2) , for any $y \in \Lambda_T(\mathrm{V})$, there exists a uniqueness element $u = \Phi^y$ such that $u \in D([0,T],\mathrm{V}) \cap L^2([0,T],\mathcal{D}(\mathrm{A}))$ and

$$du(t) + \mathrm{A}u(t)\,dt + \theta_n(\|u\|_{\Upsilon_t^V})B(u(t))\,dt = f(t)\,dt$$
$$+ \int_{\mathrm{Z}} G(y(t-),z)\widetilde{\eta}(dz,dt), \qquad u(0) = u_0.$$

Key points in the proof

Lemma

Assume that $n \in \mathbb{N}$. Assume that for all $u_0 \in \mathrm{H}$ and $f \in L^2([0,T];\mathrm{V}')$ and $y \in \Lambda_T(\mathrm{H})$, there exists an element $u = \Phi^y \in \Lambda_T(\mathrm{H})$ satisfying

$$du(t) + \mathrm{A}u(t)\,dt + \theta_n(\|u\|_{\Upsilon_t(\mathrm{H})})\mathrm{B}(u(t))\,dt$$

$$= f(t)\,dt + \int_{\mathrm{Z}} G(y(t-),z)\tilde{\eta}(dz,dt),$$

$$u(0) = u_0.$$

Then there exists a constant $C_n > 0$ such that

$$\|\Phi^{y_1} - \Phi^{y_2}\|^2_{\Lambda_T(\mathrm{H})} \le C_n T \|y_1 - y_2\|^2_{\Lambda_T(\mathrm{H})}, \quad \forall y_1, y_2 \in \Lambda_T(\mathrm{H}).$$

Remark. The above result is not true without the smoothing function θ_n.

Key points in the proof

(4). There exists a unique solution to the following SPDE

$$du_n(t) + \mathrm{A}u_n(t)\,dt + \theta_n(\|u_n\|_{\Upsilon_t^V})B(u_n(t))\,dt$$
$$= f(t)\,dt + \int_{\mathrm{Z}} G(u_n(t-), z)\widetilde{\eta}(dz, dt),$$
$$u_n(0) = u_0.$$

Well-Posedness

The Well-Posedness of strong solutions in PDE sense

Well-Posedness

Assumption 2. $G : V \times \mathrm{Z} \to V$ is a measurable mapping. There exists a constant $C > 0$ such that

(G-V1) (Lipschitz)

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{V}}^2 \nu(dz) \le C\|v_1 - v_2\|_{\mathrm{V}}^2, \quad v_1, v_2 \in \mathrm{V},$$

(G-V2) (Linear growth)

$$\int_{\mathrm{Z}} \|G(v, z)\|_{\mathrm{V}}^2 \nu(dz) \le C(1 + \|v\|_{\mathrm{V}}^2), \quad v \in \mathrm{V}.$$

(G-H2) (Linear growth)

$$\int_{\mathrm{Z}} \|G(v, z)\|_{\mathrm{H}}^2 \nu(dz) \le C(1 + \|v\|_{\mathrm{H}}^2), \ \ \forall v \in \mathrm{H}.$$

Well-Posedness

Our Second Result

Assume that **Assumption 2** holds, $u_0 \in \mathrm{V}$ and $f \in L^2_{loc}([0,\infty), H)$. Then there exists a unique $\mathbb{F}$-progressively measurable process u such that

(1) $u \in D([0,\infty), \mathrm{V}) \cap L^2_{loc}([0,\infty), \mathcal{D}(\mathrm{A}))$, $\mathbb{P}$-a.s.,

(2) the following equality in V' holds, for all $t \in [0,\infty)$, $\mathbb{P}$-a.s.:

$$\begin{aligned} u(t) &= u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t B(u(s))\,ds + \int_0^t f(s)\,ds \\ &\qquad + \int_0^t \int_{\mathrm{Z}} G(u(s-), z)\widetilde{\eta}(dz, ds). \end{aligned}$$

Well-Posedness

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(1) $u \in D([0,\infty), \mathrm{V}) \cap L^2_{loc}([0,\infty), \mathcal{D}(\mathrm{A}))$, $\mathbb{P}$-a.s.,

(2) the following equality in V' holds, for all $t \in [0,\infty)$, $\mathbb{P}$-a.s.:

$$u(t) = u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t B(u(s))\,ds + \int_0^t f(s)\,ds + \int_0^t \int_{\mathrm{Z}} G(u(s-), z)\widetilde{\eta}(dz, ds).$$

The proof is similar to the first result.

Well-Posedness

The existing results

- H. Bessaih, E. Hausenblas, P.A. Razafimandimby(2015)
  The authors considered the existence and uniqueness of solutions defined as above for stochastic hydrodynamical systems with Lévy noise, including 2-D Navier-Stokes equations.

  They assumed that the function G is globally Lipischitz in the sense that there exists $K > 0$ such that for $p = 1, 2$,

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{V}}^{2p} \nu(dz) \le K \|v_1 - v_2\|_{\mathrm{V}}^{2p}, \quad v_1, v_2 \in \mathrm{V},$$

and

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{H}}^{2p} \nu(dz) \le K \|v_1 - v_2\|_{\mathrm{H}}^{2p}, \quad v_1, v_2 \in \mathrm{H}.$$

Generalizations

Peng, Yang and Zhai, 2022, EJP

We prove similar results for stochastic 2D hydrodynamics type systems with multiplicative Lévy noises, including stochastic 2D Navier Stokes equations, 2D stochastic Magneto-Hydrodynamic equations, 2D stochastic Boussinesq model for the Bénard Convection, 2D stochastic Magnetic Bérnard problem, 3D stochastic Leray α-Model for Navier-Stokes equations, and several stochastic Shell models of turbulence.

$$\begin{aligned} &\int_{\mathrm{Z}} \|G(v_1,z)-G(v_2,z)\|_{\mathrm{H}}^2 \nu(dz) \\ &\le L_1\|v_1-v_2\|_{\mathrm{H}}^2 + L_2\|v_1-v_2\|_{\mathrm{V}}^2, \quad \forall v_1, v_2 \in \mathrm{V}, \end{aligned}$$

LDP

Part 2. Wentzell-Freidlin type large deviation principles for 2-D SNSEs driven by multiplicative Levy noise

Mumford [1] [2] :

"...... A major step in making the equation more relevant is to add a small stochastic term. Even if the size of the stochastic term goes to 0, its asymptotic effects need not. It seems fair to say that all differential equations are better models of the world when a stochastic term is added and that their classical analysis is useful only if it is stable in an appropriate sense to such perturbation"

[1] D.Mumford, The dawning of the age of stochasticity. Mathematics: frontiers and perspectives. *Amer. Math. Soc., Providence, RI*, 197–218, 2000.

[2] David Mumford: 1974 Fields medal and 2008 Wolf medal.

LDP

We consider SNSE driven by the multiplicative Lévy noise

$$\begin{cases} du^\epsilon(t) = -Au^\epsilon(t)dt - B(u^\epsilon(t))dt + f(t)dt + \sqrt{\varepsilon}\sigma(t, u^\varepsilon(t))dW(t) \\ \qquad\qquad + \epsilon \int_Z G(u^\epsilon(t-), z)\widetilde{N}^{\epsilon^{-1}}(dtdz); \\ u^\epsilon(0) = u_0 \in H. \end{cases}$$

- $W(\cdot)$ is a Wiener process.
- $N^{\epsilon^{-1}}$ is a Poisson random measure on $[0, T] \times Z$ with a σ-finite intensity measure $\epsilon^{-1}\lambda_T \otimes \nu$,
- λ_T is the Lebesgue measure on $[0, T]$ and ν is a σ-finite measure on Z.
- $\widetilde{N}^{\epsilon^{-1}}([0, t] \times O) = N^{\epsilon^{-1}}([0, t] \times O) - \epsilon^{-1}t\nu(O)$, $\forall O \in \mathcal{B}(Z)$ with $\nu(O) < \infty$, is the compensated Poisson random measure.
- σ, G are measurable mappings specified later.

LDP

As the parameter ε tends to zero, the solution u^ε will tend to the solution of the following deterministic Navier-Stokes equation

$$du^0(t) + Au^0(t)dt + B(u^0(t))dt = f(t)dt, \quad \text{with } u^0(0) = u_0 \in H.$$

LDP

Problem: We shall investigate deviations of u^ε from u^0, as ε decreases to 0.

$$Y^\varepsilon = \left(u^\varepsilon - u^0\right)/a(\varepsilon),$$

where $a(\varepsilon)$ is some deviation scale which strongly influences the asymptotic behavior of Y^ε.

(1) The case $a(\varepsilon) = 1$ provides some large deviations estimates.

LDP

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$$Y^\varepsilon = \left(u^\varepsilon - u^0\right)/a(\varepsilon),$$

where $a(\varepsilon)$ is some deviation scale which strongly influences the asymptotic behavior of Y^ε.

(1) The case $a(\varepsilon) = 1$ provides some large deviations estimates.

(2) If $a(\varepsilon)$ is identically equal to $\sqrt{\varepsilon}$, we are in the domain of the central limit theorem (CLT for short).

LDP

(3) To fill in the gap between $[a(\varepsilon) = 1]$ and $[a(\varepsilon) = \sqrt{\varepsilon}]$, it is the so-called moderate deviation principle (MDP for short).

That is when the deviation scale satisfies

$$a(\varepsilon) \to 0, \quad \varepsilon / a^2(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

LDP

Definition LDP

- $\mathcal{E} = D([0,T]; H)$, equipped with the Skorokhod topology, is a Polish space, denoted its Borel σ-field $\mathcal{B}(\mathcal{E})$.
- Rate function
  A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$, the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is a compact subset of $\mathcal{E}$.

LDP

Definition LDP

- LDP
  $(u^\varepsilon - u^0)$ obeys an LDP on $\mathcal{E}$ with rate function I, if it holds that
  (a) for each closed subset F of $\mathcal{E}$,
  $$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(u^\varepsilon - u^0 \in F\right) \leq -\inf_{x\in F} I(x);$$
  (b) for each open subset G of $\mathcal{E}$,
  $$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(u^\varepsilon - u^0 \in G\right) \geq -\inf_{x\in G} I(x).$$

We establish a Freidlin-Wentzell LDP on space $\Upsilon_T(\mathrm{V})$, here

$$\Upsilon_T(\mathrm{V}) := D([0,T],\mathrm{V}) \cap L^2([0,T],\mathcal{D}(\mathrm{A})).$$

The space $D([0,T],\mathrm{V})$ is equipped with the Skorohod topology.

Theorem

Assume that **Assumption 2** holds, $f \in L^2([0,T];\mathrm{H})$, and $u_0 \in \mathrm{V}$. Then the family $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies an LDP on Υ_T^V with the good rate function I defined by

$$I(k) := \inf\left\{L_T(g)^a : g \in \mathbb{S}, u^g = k\right\}, \quad k \in \Upsilon_T^V,$$

where for $g \in \mathbb{S}$, u^g is the unique solution of the following PDE

$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + \mathrm{B}(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t),z)(g(t,z)-1)\nu(dz),$$

$$u^g(0) = u_0.$$

[a] $L_T(g) := \int_0^T \int_{\mathrm{Z}} \Big(g(t,z)\log g(t,z) - g(t,z) + 1\Big)\nu(dz)\,dt.$

LDP

Large deviation results for 2-D SNSEs:

- Wiener Process:
  - S. Sritharan, P. Sundar, *Stochastic Process. Appl.* (2006)

    Large deviation for the two dimensional Navier-Stokes equations with multiplicative noise
  - I. Chueshov and A. Millet, *Appl. Math. Optim.* (2010)

    Stochastic 2-D Hydrodynamics Type Systems: Well Posedness and Large Deviations.
  - R. Wang, J. Zhai, T. Zhang, (JDE 2015)
    A moderate deviation principle for 2-D stochastic Navier-Stokes equations.(Also CLT)
- Levy Process:
  - T. Xu, T. Zhang, *J. Funct. Anal.* 257(2009)

    They dealt with the additive Lévy noise

LDP

- J. Zhai, T. Zhang, (*Bernoulli* 2015)
  Large deviations for 2-D stochastic Navier-Stokes equations driven by multiplicative Levy noises
- Z. Dong, J. Xiong, J. Zhai, T. Zhang, (JFA 2017)
  A moderate deviation principle for 2-D stochastic Navier-Stokes equations driven by multiplicative Levy noises
- J. Xiong, J. Zhai, (*Bernoulli* 2018)
  Large deviations for locally monotone stochastic partial differential equations driven by Levy noise
- Z. Brzeźniak, X. Peng, J. Zhai, (JEMS,2023)
  Well-posed and large deviations for 2-D Stochastic Navier-Stokes equations with jumps

THANKS!